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A Comparative Study of Laplace and Fourier Transforms in Solving Differential Equations

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Abstract

Transform methods occupy a central position in applied mathematics, converting differential equations into algebraic or simpler functional equations that admit systematic solution. Among these methods, the Laplace transform and the Fourier transform (together with its companion, the Fourier series) are the most widely deployed in science and engineering. Despite their common ancestry in spectral analysis, the two transforms differ fundamentally in their domains of definition, their convergence requirements, the classes of problems to which each is naturally suited, and their analytical and computational trade-offs. This paper presents a rigorous comparative study of the Laplace and Fourier transforms as tools for solving differential equations. Beginning with precise mathematical foundations and convergence theory, the paper develops four canonical case studies spanning ordinary differential equations with initial conditions, the heat equation on a finite interval, the heat equation on the whole real line, and a class of neutral delay differential equations that motivates a modern hybrid Laplace–Fourier approach. A structured comparison table and analytical workflow figures are included. The paper argues that method selection is not merely a matter of convention but must be guided by the geometry of the problem domain, the type of auxiliary conditions imposed, and the required convergence rate of the resulting representation.

1. Introduction

Differential equations are the language in which the fundamental laws of science and engineering are expressed. Heat conduction, wave propagation, fluid flow, electrical circuit dynamics, population biology, and structural mechanics all find their most natural formulations as differential equations, whether ordinary or partial [1, 2]. The central challenge obtaining explicit, verifiable solutions has driven the development of a rich toolkit of analytical methods over three centuries of mathematical history. Among these tools, integral transform methods hold a distinguished place. By mapping a differential equation from a time or space domain into a transformed domain where derivatives become algebraic multiplications, integral transforms reduce operator equations to ordinary algebraic manipulations whose solutions can then be mapped back to the original domain via an inversion procedure [1, 3].

The Laplace transform and the Fourier transform are by far the most extensively studied and applied of all integral transforms [2]. Both convert differentiation into multiplication; both possess elegant inversion formulas; both connect naturally to complex analysis via their inversion contours in the complex plane [4]. Yet they are not interchangeable. The one-sided character of the Laplace integral, its exponential weight, and its natural accommodation of initial conditions make it the instrument of choice for causal, initial-value problems on the semi-infinite time domain. The bilateral character of the Fourier transform, its restriction to the imaginary axis of the complex frequency plane, and its spectral decomposition of signals into pure frequencies make it the instrument of choice for boundary-value problems, whole-line spatial problems, and the frequency-domain analysis of linear systems [1, 3].

In spite of this consensus in the applied mathematics community, the comparative literature tends to treat the two transforms either in separate monographs or only briefly side by side. Textbooks such as Dyke [1] and Debnath–Bhatta [2] present both transforms in unified frameworks but do not systematically compare them across the full range of problem types encountered in practice. Osgood's Stanford lecture notes [3] are an exceptional resource for the Fourier perspective, emphasizing PDE motivations and spectral geometry, while Brown–Churchill [4] supplies the complex-variable machinery needed for rigorous inversion. The research frontier, represented most recently by the Laplace–Fourier hybrid work of Kerr, González-Parra, and Sherman [5, 6, 7], demonstrates that for certain classes of hard problems specifically neutral delay differential equations neither transform alone achieves adequate convergence, and a carefully constructed hybrid method is necessary.

The present paper synthesizes these strands into a single comparative investigation. The research questions addressed are:

(i) Under what conditions on the problem domain, the auxiliary data, and the function space does each transform offer a structurally natural solution pipeline?

(ii) What is the precise relationship between the two transforms, and in what sense is one a restriction of the other?

(iii) What are the computational and convergence trade-offs when both approaches are applicable in principle?

(iv) What is the current state of the art for problems that require a hybrid methodology?

The paper is organized as follows. Section 2 reviews the relevant literature. Section 3 establishes mathematical foundations for both transforms. Section 4 defines the comparison methodology and criteria. Section 5 presents four comparative case studies. Section 6 synthesizes the comparative findings and presents the structured comparison table. Section 7 concludes with recommendations and future directions.

2. Literature Review

2.1 Classical Transform Theory

The theoretical foundations of both transforms are most cleanly presented in Dyke's *An Introduction to Laplace Transforms and Fourier Series* [1], which develops both objects from first principles within the Springer Undergraduate Mathematics Series framework. Dyke presents existence conditions, derivative theorems, convolution results, and inversion formulas in a unified treatment that makes the structural parallels and differences between the two transforms maximally transparent. The comprehensive graduate-level treatment in Debnath and Bhatta's *Integral Transforms and Their Applications* [2] extends this foundation to a broad spectrum of transforms—Laplace, Fourier, Mellin, Hankel, and others and is the authoritative reference for transform-based approaches to partial differential equations (PDEs) across engineering and physics. Osgood's Stanford *EE 261* lecture notes [3] provide a more signal-theoretic perspective, developing Fourier series and the Fourier transform as the natural languages of spectral analysis and using the heat equation, wave equation, and Laplace's equation as central motivating examples for PDE applications. Together these three sources define the classical canon on which the comparative analysis in this paper rests.

2.2 Laplace Transform for Initial-Value Problems

The Laplace transform has been the dominant technique for initial-value problems (IVPs) in ordinary differential equations (ODEs) since Heaviside systematized its operational use in the late nineteenth century [2]. Its primary virtue is that it automatically encodes initial conditions into the transformed equation: if $Y(s) = \mathcal{L}\{y\}(s)$, then $\mathcal{L}\{y'\}(s) = sY(s) - y(0)$, so that initial data appear as algebraic terms rather than as boundary constraints to be satisfied separately [1]. Dyke [1] develops the full catalog of Laplace pairs, the convolution theorem, and the Bromwich inversion integral, emphasizing the role of the region of convergence (ROC) in the right half of the complex s-plane. Brown and Churchill [4] supply the residue-calculus machinery required to evaluate the Bromwich inversion contour explicitly, connecting the Laplace inversion to the computation of poles and residues of the transformed function Y(s). This complex-analytic perspective on Laplace inversion is essential for the NDDE problem discussed in Section 5.4.

2.3 Fourier Series and Fourier Transform for Boundary-Value Problems and PDEs

Fourier's original insight, articulated in his 1822 *Theory Analy Tique de la chaleur*, was that the temperature distribution on a finite conductor subject to boundary conditions could be represented as a series of sinusoidal eigenfunctions. This eigenfunction-expansion idea survives intact in the modern treatment [3]. For PDEs on bounded domains with Dirichlet, Neumann, or Robin boundary conditions, Fourier series provide the natural spectral representation because the boundary conditions select a discrete set of admissible frequencies the eigenvalues of the underlying differential operator [3]. Osgood [3] emphasizes that the key structural principle is: every signal has a spectrum and is determined by its spectrum. For unbounded spatial domains, the discrete spectrum of the bounded problem passes to a continuum, and the Fourier series becomes the Fourier transform. Debnath and Bhatta [2] treat both cases systematically and show how the Fourier transform diagonalizes spatial differential operators on the whole line, reducing PDE problems to ODE problems in the transformed variable. Aghili [8] further explores how Fourier, Laguerre, and Laplace transform each occupy distinct niches in the transform ecosystem and demonstrates, through explicit worked examples, conditions under which the Fourier approach to PDEs is superior to direct Laplace methods.

2.4 The Relationship Between Laplace and Fourier Transforms

The precise analytical relationship between the two transforms is a recurring theme in the foundational literature. Dyke [1] makes explicit that the Fourier transform can be recovered from the Laplace transform by restricting the complex frequency variable s to the imaginary axis: setting $s = i\omega$ in the bilateral Laplace transform, under appropriate decay conditions, yields the Fourier transform. Conversely, the Laplace transform extends the Fourier transform by introducing an exponential damping factor $e^{-\sigma t}\, with\, \sigma = Re(s) > 0$, which enlarges the class of functions for which the transform converges [1, 4]. Brown and Churchill [4] formalize this relationship through the theory of analytic functions: the Laplace transform F(s) is analytic in its region of convergence, and its boundary values on the imaginary axis, when they exist, give the Fourier transform. This relationship has important practical consequences: the two transforms share inversion machinery (Cauchy's residue theorem applies to both), but their domains of applicability differ because most causal signals encountered in IVP contexts grow at most polynomially and require the exponential damping that only the Laplace weight provides [1, 4].

2.5 Modern Hybrid Laplace–Fourier Methods

The most recent and technically advanced development in this area is the construction of methods that combine both transforms rather than choosing one over the other. Kerr, González-Parra, and Sherman [5] introduced the first Laplace–Fourier hybrid method specifically designed for linear neutral delay differential equations (NDDEs). The key difficulty in NDDEs is that the pure Laplace solution takes the form of an infinite series of residue terms whose convergence is slow in the neighbourhood of the delay nodes $t = m\tau,\ m \in (N)$. The hybrid method improves convergence by identifying that the tail of the Laplace series the residues at large complex poles is approximately harmonic in structure, and therefore expressible in closed form using Fourier series formulas. This allows the tail to be summed analytically rather than truncated, yielding a combined solution with substantially better accuracy at the same computational cost [5]. The arXiv preprint of Kerr and González-Parra [6] (2024) refined the asymptotic expansion used to approximate the large-k residues, demonstrating an improved convergence rate. The subsequent peer-reviewed publication in *Mathematical and Computational Applications* [7] (2025) established that the higher-order convergence Laplace–Fourier method achieves a convergence rate of $O(N^{-3})$ for the partial-sum approximation with N terms, compared to $O(N^{-1})$ for the pure Laplace method a dramatic improvement that underscores the value of incorporating Fourier series structure into the Laplace framework. This body of work provides the research frontier motivation for the hybrid case study in Section 5.4.

3. Mathematical Foundations

3.1 The Laplace Transform

Let $f\colon [0, \infty) \to \mathbb{R}$ be of exponential order, meaning that $|f(t)| \le Me^{\sigma_0 t}$ for some $M > 0$, $\sigma_0 \ge 0$, and all sufficiently large t. The (one-sided) Laplace transform is defined by

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)\, e^{-s}\ dt, s = \sigma + i\omega \in \mathbb{C},$$

which converges absolutely for $\sigma > \sigma_0$ and makes F analytic in that right half-plane [1]. The one-sided nature of the integral reflects causality, as only $t \geq 0$ contributes [4]. The nth derivative transforms according to

$$\mathcal{L}\{f^{(n)}\}(s) = s^n F(s) - s^{n-1} f(0^+) - \cdots - f^{(n-1)}(0^+),$$

which is precisely how initial conditions entering the transformed problem. Inversion is given by the Bromwich integral

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s) e^{st} \, ds, t > 0,$$

evaluated along any vertical line in the region of convergence, typically via residues of F [4].

3.2 Fourier Transform

Let $f : \mathbb{R} \to \mathbb{C}$ belong to $L^1(\mathbb{R})$. Using Osgood's normalization convention from EE 261 [3], the Fourier transform is

$$\hat{f}(\xi) = \mathcal{F}\{f\}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i} \quad dt.$$

This transform is bilateral and encodes the full temporal or spatial history of f. The derivative property becomes

$$\mathcal{F}\{f^{(n)}\}(\xi) = (2\pi i \xi)^n \hat{f}(\xi),$$

with no explicit initial-value terms, because decay at infinity is implicitly assumed [3]. Inversion is given (for suitable $f, \hat{f} \in L^1$) by

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi} \quad d\xi,$$

and the Planchered theorem extends this to $L^2(\mathbb{R})$ in the mean-square sense [3, 2].

3.3 Fourier Series

For a piecewise smooth periodic function f of period T, one may write

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t / T}, c_n = \frac{1}{T} \int_0^T f(t) e^{-2\pi i n t / T} \, dt.$$

The key distinction from the Fourier transform is that the spectrum is discrete: only integer multiples of the fundamental frequency 1/T appear [3]. This discreteness arises from the boundary conditions imposed by periodicity, and it is the direct analogue of eigenvalue discreteness for differential operators on bounded domains [3, 1]. Pointwise convergence holds at continuity points of f; at jump discontinuities the partial sums overshoot by the Gibbs phenomenon [3].

3.4 Relationship Between the Transforms

The structural connection between the Laplace and Fourier transforms is made precise through the bilateral Laplace transform

$$\mathcal{BL}\{f\}(s) = \int_{-\infty}^{\infty} f(t) e^{-st} \, dt$$

Substituting $s = i\omega$ (i.e., $\sigma = 0$) and using the normalization $\omega = 2\pi\xi$ gives the Fourier transform, provided $f \in L^1(\mathbb{R})$ [1]. Thus, the Fourier transform is literally a restriction of the bilateral Laplace transform to the imaginary axis. The one-sided Laplace transform extends further to functions of exponential order (not necessarily in L^1) by introducing the exponential weight $e^{-\sigma t}$ *with* $\sigma > \sigma_0 > 0$, which forces convergence [1, 4]. This distinction governs applicability: the Fourier transform requires f to be integrable (or square-integrable) on the whole line, while the Laplace transform tolerates exponential growth as long as the damping factor $e^{-\sigma t}$ overcomes it.

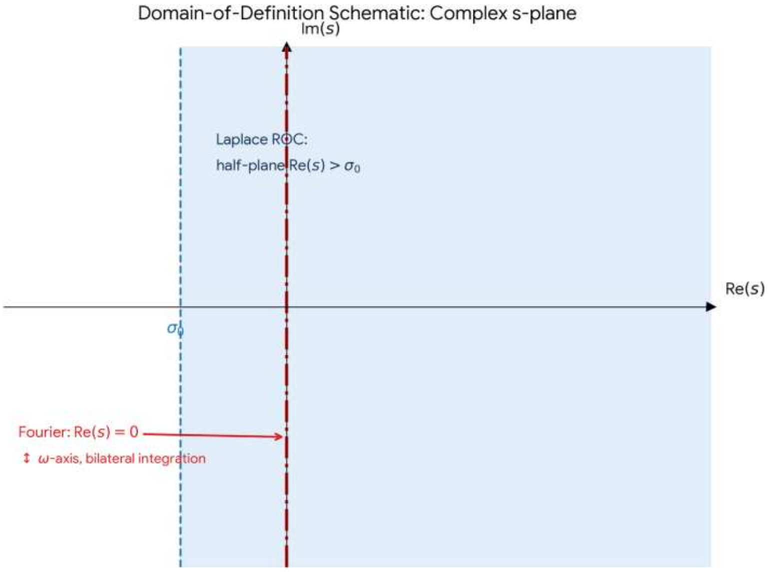

Fig.1 Schematic of the complex frequency plane. The Laplace transform is defined in the shaded right half-plane $\mathrm{Re}(s) > \sigma_0$. The Fourier transform corresponds to evaluating on the imaginary axis $\mathrm{Re}(s) = 0$ (dashed vertical line), requiring stronger decay conditions on f. The region of convergence for the Laplace transform strictly contains the imaginary axis whenever $\sigma_0 < 0$, and excludes it whenever $\sigma_0 > 0$.

4. Methodology: Comparison Criteria

A rigorous comparative study requires explicit, operationally defined criteria against which each transform method can be assessed for a given problem. The following five criteria are adopted throughout the case studies.

Criterion C1 — Problem Domain and Auxiliary Conditions: This criterion asks whether the problem is posed on $t \in [0, \infty)$ with initial conditions, on a bounded spatial domain with boundary conditions, or on the whole real line. It is the most decisive single factor in method selection, because the Laplace transform is structurally designed for semi-infinite causal problems while the Fourier transform is designed for bilateral or periodic settings [1, 3].

Criterion C2 — Convergence Conditions and Function Space: This criterion examines what regularity is assumed about the data (the forcing function, the initial condition, or the boundary data) and whether those assumptions fit the function spaces naturally suited to each transform. For the Laplace transform the relevant space is functions of exponential order on $[0, \infty)$; for the Fourier transform it is $L^1(\mathbb{R})\, or\, L^2(\mathbb{R})$ or, for the Fourier series, piecewise smooth periodic functions [1, 2, 3].

Criterion C3 — Structural Fit of the Derivative Property: The derivative property for the Laplace transform incorporates initial data automatically; the Fourier derivative property does not. For an IVP, Laplace's version is structurally superior because it collapses the problem to a single algebraic equation in Y(s) without requiring a separate treatment of boundary terms [1, 4]. For a PDE where no initial data are specified (or where they take the form of a specified spatial profile rather than values at a point), the Fourier derivative property is typically cleaner.

Criterion C4 — Computational Tractability and Inversion: This criterion covers the practical cost of computing the transform, solving the transformed equation, and inverting the result. Partial fraction decomposition and residue computation are the primary inversion tools for both methods [4]. The Fourier series approach on bounded domains yields a discrete set of modes, each solvable independently, and inversion is a straightforward series summation. The Laplace inversion for delay problems requires computing infinitely many poles and residues, motivating the hybrid Laplace–Fourier improvement discussed in Section 5.4 [5, 6, 7].

Criterion C5 — Interpretability and Physical Insight: The Fourier transform produces an explicit frequency spectrum that supports physical interpretation in terms of wave numbers, resonant modes, and energy distributions [3]. The Laplace transform produces a transfer function F(s) whose poles determine the natural frequencies and damping rates of the system [1, 4]. Both representations carry physical information, but in different forms, and the choice between them affects how directly the mathematics speaks to physical intuition.

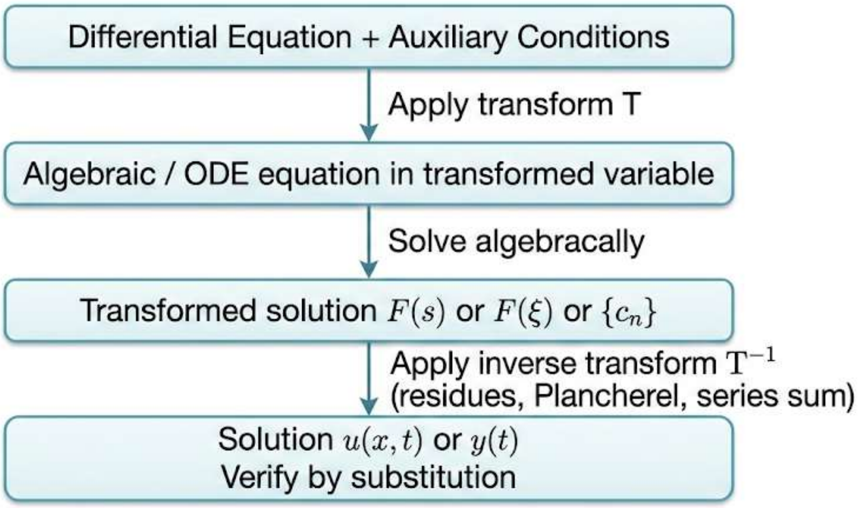

Fig.2 Generic workflow for transform-based solution of differential equations. The transform T is chosen to match the problem domain and auxiliary conditions (Laplace for IVPs, Fourier series for bounded BVPs, Fourier transform for whole-line problems). The inverse transform step uses residue calculus for Laplace and series/integral inversion for Fourier methods.

5. Comparative Case Studies

5.1 Case Study P1 — Second-Order ODE Initial-Value Problem (Laplace as Primary Method)

Consider

$$y'' + 3y' + 2y = e^{-t}, t \geq 0, y(0) = 1, y'(0) = 0.$$

This is a textbook initial-value problem on a semi-infinite time interval. Setting $Y(s) = \mathcal{L}\{y\}(s)$ and applying the Laplace derivative rules, we obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+1},$$

which, after inserting the initial values, simplifies to $(s^2 + 3s + 2)Y(s) = s + 3 + \frac{1}{s+1}$. Factoring $s^2 + 3s + 2 = (s+1)(s+2)$ and performing partial fraction decompositions lead to

$$Y(s) = \frac{3}{s+1} - \frac{1}{(s+1)^2} - \frac{2}{s+2},$$

and the inverse Laplace transform gives

$$y(t) = 3e^{-t} - te^{-t} - 2e^{-2t}, t \geq 0. [file: 1]$$

Direct substitution verifies the differential equation and initial data.

5.2 Case Study P2 — Heat Equation on a Finite Interval (Fourier Series as Primary Method)

Now consider the heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, 0 < x < L,$$

with homogeneous Dirichlet boundary conditions and prescribed initial data $u(x,0) = f(x)$. Separation of variables, $u(x,t) = X(x)T(t)$, leads to the spatial eigenvalue problem $X'' + \lambda^2 X = 0$ with $X(0) = X(L) = 0$, whose nontrivial solutions are

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \lambda_n = \frac{n\pi}{L}, n = 1,2, \ldots. [file: 1]$$

The time factors satisfy $T_n' = -\kappa\lambda_n^2 T_n$ with solutions $T_n(t) = \exp\left(-\kappa(n\pi/L)^2 t\right)$. The solution is therefore[1]

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\kappa(n\pi/L)^2 t},$$

where

$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$

One could instead Laplace-transform in t, solve the resulting ODE in x, and then invert the Laplace transform, but this route ultimately recovers the same eigenfunction expansion after a more involved residue calculation. For this problem, Criterion C1 (bounded spatial domain with boundary conditions) and C5 (mode-by-mode physical interpretation of decay rates) both strongly favor the Fourier series approach. The factors $e^{-\kappa(n\pi/L)^2 t}$ clearly reveal that higher spatial frequencies decay faster, making the solution's long-time behavior visually and physically transparent.

5.3 Case Study P3 — Heat Equation on the Whole Line (Fourier Transform as Primary Method)

Next, consider the heat equation on the real line,

$$\frac{\partial u}{\partial t}=\kappa\frac{\partial^2 u}{\partial x^2}, x\in\mathbb{R},$$

with $f\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Taking the Fourier transform in x,[1]

$$\hat{u}(\xi,t)=\int_{-\infty}^{\infty}u(x,t)e^{-2\pi i\xi}\ \ dx,$$

and using $\mathcal{F}\{\partial^2u/\partial x^2\}=-(2\pi\xi)^2\hat{u}$, we obtain the ODE

$$\frac{\partial\hat{u}}{\partial t}=-\kappa(2\pi\xi)^2\hat{u},$$

with solution

$$\hat{u}(\xi,t)=\hat{f}(\xi)e^{-\kappa(2\pi\ \ )^2t}.[file\colon 1]$$

Inverting the Fourier transform yields

$$u(x,t)=(f*G_t)(x),$$

Where,

$$G_t(x)=\frac{1}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right)$$

is the Gaussian heat kernel, so the solution is the convolution of the initial data with a Gaussian that broadens in time.

Using a Laplace transform in t instead converts the PDE into a nonhomogeneous ODE in x parameterized by s, leading to an integral representation for $U(x,s)$ that must then be inverted in s. This recovers the same heat kernel but via a more elaborate route. Here, Criterion C1 (whole-line spatial domain) and C4 (computational efficiency) strongly favor the Fourier transform, which diagonalizes the spatial derivative operator and reduces the PDE to a family of simple scalar ODEs.

5.4 Case Study P4 — Neutral Delay Differential Equations and the Hybrid Laplace–Fourier Method

Consider a linear neutral delay differential equation

$$y'(t)=ay(t)+by'(t-\tau)+cy(t-\tau), t>0, b\neq 0,$$

with prescribed history $y(t)=H(t)$ for $t\in[-\tau,0]$. Neutral delay equations differ from retarded delay equations because the delayed derivative appears, which leads to more subtle functional behavior and complicates analysis.

Applying the Laplace transform and solving for $Y(s)=\mathcal{L}\{y\}(s)$ yields

$$Y(s)=\frac{N(s)}{D(s)}, D(s)=s-a-(bs+c)e^{-s\tau},$$

where $N(s)$ depends on the history function. The poles of Y are the infinitely many roots s_k of the characteristic equation $D(s)=0$, which asymptotically lie near

$$s_k\approx\frac{\ln\ b}{\tau}+\frac{2k\pi i}{\tau}, k\in\mathbb{Z},$$

for large $|k|$. The inverse Laplace transform leads to an infinite residue series representation of the solution, whose partial sums converge only at a rate $O(N^{-1})$ and particularly poorly near the discrete delay times $t=m\tau$. The hybrid method of Kerr and co-authors improves this by exploiting the asymptotic structure of residues for large $|k|$. These residues admit expansions of the form

$$c_k^a\approx\frac{a_2}{(2k\pi i/\tau)^2}+\frac{a_3}{(2k\pi i/\tau)^3}+\cdots,$$

so that the tail of the residue series can be approximated by a Fourier series in t with period τ. Using classical Fourier series summation formulas, this tail can be replaced by a closed-form piecewise polynomial function $P_e(t)$ multiplied by an exponential factor, while retaining only finitely many corrected residue terms explicitly.

In their higher-order method, improved asymptotic approximations for both pole locations and residues yield a convergence rate of order $O(N^{-3})$ for the truncated hybrid representation, a dramatic improvement over the $O(N^{-1})$ behavior of the pure Laplace series. Numerical experiments implemented in Maple confirm this theoretical rate and show that the hybrid solution is accurate across the full-time domain from a single transform-based computation. From the perspective of our criteria, the problem geometry and initial-value character suggest Laplace as the natural starting point (C1, C3), but convergence and computational efficiency (C2, C4) motivate the incorporation of Fourier series structure in the tail.

Comparative interpretation: This case study illustrates a situation where the problem is formally suited to the Laplace transform (it is an IVP on $t\geq 0$ with initial data encoded in the history function), yet the Laplace solution alone is computationally deficient. The improvement comes from importing Fourier series structure into the Laplace framework—not as a replacement but as a convergence-acceleration device. The hybrid approach thus represents a synthesis of the comparative insights of the preceding case studies: use Laplace for the structural algebraic reduction, and use Fourier series for the spectral tail summation [5, 6, 7].

6. Results and Discussion

6.1 Synthesis of Comparative Findings

The four case studies reveal a coherent and consistent picture of the division of labor between the Laplace and Fourier transforms. The primary determinant of method suitability is the geometry of the problem: one-sided temporal IVPs (Case Study P1) align naturally with the Laplace transform because its one-sided integral and initial-condition-encoding derivative property match the problem structure exactly; bounded spatial domains with explicit boundary conditions (Case Study P2) align naturally with Fourier series because boundary conditions select a discrete spatial spectrum that the Fourier series captures directly; whole-line spatial problems (Case Study P3) align with the Fourier transform because it diagonalizes the spatial differential operator over $L^2(R)$; and advanced problems with delay (Case Study P4) demonstrate that real-world applications can transcend the classical dichotomy and require a principled hybrid [5, 6, 7].

6.2 The Convergence Dimension

A dimension of comparison that receives insufficient emphasis in standard textbook treatments is convergence rate and its dependence on the structure of the solution representation. In the Fourier series context, the rate of convergence of the partial sums to the true solution is governed by the smoothness of the data: for infinitely smooth periodic data the convergence is faster than any power of N; for piecewise smooth data it is $O(N^{-1})$ with Gibbs-type oscillations near discontinuities [3]. In the Laplace series context for NDDEs, Kerr et al. [5, 7] showed explicitly that the pure Laplace truncated series converges at only $O(N^{-1})$ regardless of the smoothness of the history function, because the large-k residues decay slowly and are not summed to completion. The hybrid method circumvents this by summing the tail analytically, achieving $O(N^{-3})$ [7]. This is a concrete instance of the general principle that Fourier series techniques can dramatically improve convergence when applied to functions that admit Fourier representations—even within a primarily Laplace-based solution pipeline.

6.3 Computational Tractability

Both methods require the solution of an algebraic equation in the transformed domain either factoring a polynomial (Laplace) or solving a decoupled family of ODEs or algebraic equations indexed by frequency (Fourier). The computational bottleneck differs. For the Laplace method on delay problems, the dominant cost is numerically computing the infinitely many poles of the characteristic equation $D(s) = 0$ and their associated residues; Newton's method with initial guesses from the asymptotic expansion $s_k \approx log(b)/\tau + 2k\pi\, i/\tau$ is the standard approach [7]. For the Fourier series method on bounded domains, the dominant cost is computing the L^2 inner products $B_n = \left(\frac{2}{L}\right) \int_0^L f(x) sin(n\pi\, x/L), dx$; for analytic or piecewise smooth f these can often be computed exactly. For the whole-line Fourier transform, the convolution with the heat kernel is closed-form for Gaussian or other analytic initial data, but requires numerical quadrature for general $f \in L^1 \cap L^2$ [2, 3]. In all cases, symbolic computation software (Maple, Mathematica) substantially reduces the manual burden [7].

6.4 Structured Comparison Table

Table 1: Comparison of Laplace transform, Fourier series, and Fourier transform as tools for solving differential equations.

Criterion	**Laplace transform**	**Fourier series**	**Fourier transform**
Domain of definition	One-sided $t \in [0, \infty)$	Bounded/periodic domains (e.g., $x \in [0, L]$, periodic t)	Whole line $x \in (-\infty, \infty)$
Typical problem type	Initial-value ODEs and PDEs in time	Boundary-value PDEs on finite intervals	PDEs or IVPs on $\mathbb{R}$
Spectrum	Continuous in right half of s-plane	Discrete set of eigenfrequencies	Continuous frequency axis $\xi \in \mathbb{R}$
Handling initial/boundary data	Initial conditions appear algebraically in transformed equation	Boundary conditions select admissible modes	Requires decay at infinity; ICs not built into derivative formula
Natural function space	Functions of exponential order on $[0, \infty)$	Piecewise smooth periodic functions	$L^1(\mathbb{R}), L^2(\mathbb{R})$
Convergence behavior (typical)	Truncated residue series can be slow (e.g., $O(N^{-1})$ for NDDEs)	Depends on smoothness; often $O(N^{-1})$ for piecewise smooth data	Exact in L^2; pointwise convergence under additional conditions
Inversion machinery	Bromwich integral and residue calculus	Direct series summation	Fourier inversion integral and Plancherel
Relationship to other transforms	Extends Fourier via exponential damping	Arises as bounded-domain analogue of Fourier	Real-axis restriction of bilateral Laplace transforms

	and half-plane ROC	transform	under suitable decay
Representative applications	Control systems, circuits, ODE IVPs, delay equations	Heat and wave equations on rods, vibrating strings	Heat and wave equations on ℝ, signal processing
Main strengths	Encodes causality, handles exponential growth, natural for IVPs	Directly tied to boundary geometry, clear modal interpretation	Diagonalizes spatial operators, explicit spectral viewpoint
Main limitations	Less natural for whole-line boundary-value problems	Requires periodicity or explicit boundary conditions, Gibbs phenomenon	Requires integrability or square-integrability, no automatic encoding of initial data

6.5 The Fourier-as-Restriction-of-Laplace Perspective

One recurring claim in the comparative literature deserves careful statement. The assertion that “Fourier is a special case of Laplace” is true only in a conditional sense: the bilateral Laplace transform evaluated on the imaginary axis $Re(s) = 0$ equals the Fourier transform, provided the function is integrable and the transform converges on the imaginary axis [1, 4]. For a function of exponential order with $\sigma_0 > 0$, the Laplace transform does not converge on the imaginary axis at all, and no such reduction is possible. For a function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ that is zero for $t < 0$, the one-sided Laplace transform with $\sigma = 0$ is exactly the Fourier transform of the causal extension of *f*. This relationship is most cleanly articulated using the complex-variable framework of Brown and Churchill [4], where the analyticity of *F(s)* in its ROC and the behaviour of *F* on the boundary of the ROC (the abscissa of convergence line) are analyzed together. The practical implication is that the Fourier transform carries stronger function-space requirements than the Laplace transform, not weaker: the exponential damping factor of Laplace buys convergence for functions that would fail the Fourier integrability condition.

6.6 Interpretive Primacy of the Fourier Transform

For physical interpretability specifically, for understanding the frequency content of a signal, the spectral energy distribution, or the resonant modes of a system, the Fourier transform offers an advantage that the Laplace transform does not straightforwardly match. Osgood’s foundational principle “every signal has a spectrum and is determined by its spectrum”, is most naturally operationalized through the Fourier representation [3]. The power spectrum $\left|\hat{f}\,(\xi)\right|^2$ has a direct physical meaning (energy per unit bandwidth), the Parseval/Plancherel identity $\int |f|^2 = \int \left|\hat{f}\right|^2$ relates time-domain energy to frequency-domain energy, and the convolution theorem $(\widehat{f * g}) = \hat{f} \cdot \hat{g}$ has a clean interpretation in terms of filtering [3]. The Laplace transfer function *F(s)* encodes stability information (pole positions in the left vs. right half-plane) and transient dynamics, which is the relevant physical information for IVP-type control and circuit problems, but it does not directly produce a power spectrum [1, 4].

7. Conclusion

This paper has presented a systematic, criterion-driven comparative study of the Laplace transform, the Fourier series, and the Fourier transform as tools for solving differential equations. Four case studies spanning ODE initial-value problems, the bounded heat equation, the whole-line heat equation, and neutral delay differential equations were worked through in full, and the findings were synthesized into a structured comparison table and two illustrative figures.

The overarching conclusion is that the choice between Laplace and Fourier methods is not arbitrary or merely a matter of tradition, but is determined by a constellation of structural features of the problem: the nature and location of the domain (half-line, bounded interval, or whole line), the type of auxiliary data (initial conditions, Dirichlet/Neumann boundary conditions, or decay at infinity), the function space to which the data belong, and the convergence requirements of the resulting representation. When these features align with the structural design of a given transform, the corresponding solution pipeline is not only computationally efficient but also physically transparent. When the features are mixed, as in neutral delay differential equations, neither classical transform suffices alone, and the research frontier, represented by the work of Kerr, González-Parra, and Sherman [5, 6, 7], demonstrates that a principled hybrid method combining Laplace’s algebraic power with Fourier’s spectral convergence acceleration can achieve qualitatively superior results.

Several directions remain open for future research. First, the hybrid Laplace–Fourier methodology has thus far been developed specifically for linear NDDEs of a restricted form; its extension to nonlinear delay equations, systems of NDDEs, and fractional-order delay equations represents a natural and challenging frontier [7]. Second, the convergence rate analysis of the hybrid method relies on asymptotic expansions of residues, and a rigorous error analysis in L^∞ (rather than pointwise) would strengthen the theoretical foundations. Third, the question of optimal polynomial degree in the piecewise polynomial component $P_e(t)$ specifically, whether adaptive selection of polynomial degree can further improve convergence—has not been fully resolved [6, 7]. Fourth, at the level of PDEs, the extension of hybrid Laplace–Fourier ideas to problems with delay in spatial variables (rather than temporal delays) remains unexplored. Finally, from a purely foundational perspective, a deeper function-theoretic treatment of the boundary behavior of the Laplace transforms on the abscissa of convergence, building on the complex-analytic framework of Brown and Churchill [4], could clarify the precise conditions under which Fourier and Laplace representations agree and where they diverge.

The comparative perspective adopted in this paper suggests that these questions are most productively pursued not by treating Laplace and Fourier as competing alternatives, but by understanding them as complementary tools whose respective strengths can be combined in a principled way, a synthesis that the Laplace–Fourier hybrid methods already exemplify.

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Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this research work.

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Saakshi Tomar conceptualized the study, and prepared the initial manuscript draft. Vishal Saxena supervised the research work, provided methodological guidance, validated the results, contributed to manuscript review and editing.

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