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Some Results Concerned to Generalized Functions of Fractional Calculus

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Abstract

In the present paper, the author introduced the functions $K(c,\upsilon,p,q,x)$ and $K(c,-\mu,p,q,x)$ in terms of generalized M-series and its properties by using fractional calculus

1. Introduction

The function which is introduced and studied by Mittag-Leffler [3,4] in terms of the power series given below

$$E_{\alpha}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\alpha n+1)},\quad (\alpha>0) \tag{1.1}$$

A generalization of this series in the following form

$$E_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\alpha n+\beta)},\quad (\alpha,\beta>0) \tag{1.2}$$
is given by Wiman [2].

The generalized M-Series [8] is given by

$$ {}_{p}M_{q}^{\alpha,\beta}(a_1,\dots,a_p;b_1,\dots,b_q;x)={}_{p}M_{q}^{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{x^{n}}{\Gamma(\alpha n+\beta)} \tag{1.3}$$

where $\alpha,\beta\in C, R(\alpha)>0$ and $(a_i)_n(i=1,2,\dots,p)$ and $(b_j)_n\ (j=1,2,\dots,q)$ are the Pochhammer symbols. Further details of this series are given by [8].
The Riemann-Liouville operator of fractional integral of order υ is given by

$$I_x^{\upsilon}\{f(x)\}=\frac{1}{\Gamma(\upsilon)}\int_0^x (x-t)^{\upsilon-1}f(t)dt \tag{1.4}$$

provided that the integral exists.

The Riemann-Liouville operator of fractional derivative of order υ is defined [1,5,6,7] in the following form

$$D_x^{\upsilon}\{f(x)\}=\frac{1}{\Gamma(\upsilon)}\frac{d^n}{dx^n}\int_0^x\frac{f(t)}{(x-t)^{\upsilon+n-1}}dt,(n-1<\upsilon<n) \tag{1.5}$$

provided that the integral exists.

2. Fractional Calculus Operators and Generalized M-Series

Let

$$f(x)=\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{(cx)^n}{n!} \tag{2.1}$$

where c is an arbitrary constant.
The fractional integral operator of order ν is given by

$$I_x^{\upsilon}\{f(x)\}=\frac{1}{\Gamma(\upsilon)}\int_0^x (x-\tau)^{\upsilon-1}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{(c\tau)^n}{n!}d\tau$$

$$=\frac{1}{\Gamma(\upsilon)}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{c^n}{n!}\int_0^x (x-\tau)^{\upsilon-1}\tau^n d\tau$$

$$=x^{\upsilon}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{(cx)^n}{\Gamma(\upsilon+n+1)}$$

By using (1.3), the above equation can be written as

$$=x^{\upsilon}\,{}_{p}M_{q}^{1,\upsilon+1}(ct) \tag{2.2}$$

The author introduced a new function which is given below

$$K(c,\upsilon,p,q,x)=x^{\upsilon}\,{}_{p}M_{q}^{1,\upsilon+1}(cx) \tag{2.3}$$

Now, the fractional differential operator of order μ is given by

$$D_x^{\mu}\{f(x)\}=D^k\{I_x^{k-\mu}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{(cx)^n}{n!}\}$$

On simplifying, we arrive at

$$= D^k \{x^{k-\mu} \sum_{n=0}^{\infty} \frac{(a_1)_n ... (a_p)_n}{(b_1)_n ... (b_q)_n} \frac{(cx)^n}{\Gamma(k-\mu+n+1)}$$

$$= x^{-\mu} \sum_{n=0}^{\infty} \frac{(a_1)_n ... (a_p)_n}{(b_1)_n ... (b_q)_n} \frac{(cx)^n}{\Gamma(n+1-\mu)}$$

Again, by using (1.3), the above equation can be written as

$$= x^{-\mu} \, {}_pM_q^{1,1-\mu}(ct) \tag{2.4}$$

3. Properties of the Function $K(c,\upsilon,p,q,x)$:

Theorem 3.1 If c is an arbitrary constant then

$$I_x^{\lambda} K(c,\upsilon,p,q,x) = K(c,\lambda+\upsilon,p,q,x) \tag{3.1}$$

Proof:

From the definition of the fractional integral, we have

$$I_x^{\lambda} K(c,\upsilon,p,q,x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-\tau)^{\lambda-1} K(c,\upsilon,p,q,\tau) d\tau \tag{3.2}$$

Using (2.3), it reduces to

$$= \frac{1}{\Gamma(\lambda)} \int_0^x (x-\tau)^{\lambda-1} \tau^{\upsilon} \sum_{n=0}^{\infty} \frac{(a_1)_n ... (a_p)_n}{(b_1)_n ... (b_q)_n} \frac{(c\tau)^n}{\Gamma(\upsilon+n+1)} d\tau$$

On substituting $\tau = zx$, it yields

$$= \frac{1}{\Gamma(\lambda)} x^{\lambda+\upsilon} \sum_{n=0}^{\infty} \frac{(a_1)_n ... (a_p)_n}{(b_1)_n ... (b_q)_n} \frac{(cx)^n}{\Gamma(\upsilon+n+1)} \int_0^1 (1-z)^{\lambda-1} z^{k+\upsilon} dz \tag{3.3}$$

On simplifying and using (2.3), we arrive at

$$I_x^{\lambda} K(c,\upsilon,p,q,x) = K(c,\lambda+\upsilon,p,q,x) \tag{3.4}$$

Hence proved.

Theorem 3.2 If c is an arbitrary constant then

$$D_x^{\lambda} K(c,\upsilon,p,q,x) = K(c,\upsilon-\lambda,p,q,x)$$

Proof: By the definition of the fractional differential, we get

$$D_x^{\lambda} K(c,\upsilon,p,q,x) = D^k \{I_x^{k-\lambda} K(c,\upsilon,p,q,x)\}$$

$$= D^k \{x^{k+\upsilon-\lambda} \, {}_pM_q^{1,k+\upsilon-\lambda+1}(cx)\}$$

Applying (2.3), we arrive at

$$D_x^{\lambda} K(c,\upsilon,p,q,x) = K(c,\upsilon-\lambda,p,q,x)$$

This proves theorem (3.2).

Conflict of Interest

The authors declare that there is no conflict of interest related to this research work.

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