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Advancements in Recurrence Relations Associated with the Generalized M-Series

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Abstract

This paper deals with the recurrence relations of generalized Mittag-Leffler type function namely generalized M-series. The results derived in this paper are the extensions of the results derived earlier by some authors. The special cases are also considered.

1. Introduction

The classical calculus was independently discovered in 17^{th} century by Isaac Newton and Gottfried Wilhelm Leibnitz. The question raised by Leibnitz for the existence of fractional derivative of order, half was an ongoing topic amongst mathematicians for more than three hundred years, consequently several aspects of fractional calculus were developed and studied. During last decade applied mathematicians and physicists found the fractional calculus operators to be very useful in a variety of fields such as quantitative biology, electro chemistry, scattering theory, transport theory, probability, elasticity, control theory, potential theory, signal processing, image processing, diffusion theory, kinetic theory, heat transfer theory and circuit theory etc. The fractional calculus operators also occur widely in technical problems associated with transmission lines and the theory of compressional shock waves. The fractional calculus is a generalization of ordinary differentiation to non-integer case. In other words, the fractional calculus operators deal with integrals and derivatives of arbitrary (i.e. real or complex) order. The name “fractional calculus” is actually a misnomer; the designation, “integration and differentiation of arbitrary order” is more appropriate. The first accurate use of a derivative of non-integer order is due to the French mathematician S. F. Lacroix [22] in 1819 who expressed the derivative of non-integer order ½ in terms of Legendre’s factorial symbol Γ. Starting, with a function $y = x$, m, Lacroix expressed it as follows Replacing with ½ and putting $m = 1$, he obtained the derivative of order 1/2 of the function x.

The credit of first application of fractional calculus goes to Abel’s [11] who employed it in the solution of an integral equation which emerged in the formulation of the tautochrone problem of finding the shape of a frictionless wire lying in a vertical plane such that the time of slide of a bead placed on the wire to the lowest point of the wire is the same regardless of position of the bead on the wire.

The importance of special functions as a device of mathematical analysis is well known to the scientist, mathematician and engineers dealing with the practical applications of differential equations. The solution of various problems from the heat conduction, electromagnetic waves,

fluid mechanics, quantum mechanics, kinetic equations and diffusion equations etc. lead obligatory to using the special function. Special functions arise as a solution of some basic ordinary differential equations and solving partial differential equations by means of separation of variable method. The verity of the nature of the methods leading to special functions stimulated the increasing of the number of special functions used in applications.

The Mittag-Leffler function introduced by Mittag Leffler [5] in 1903 is defined as

$$E_{\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + 1)} \tag{1}$$

where $\alpha, \beta \in C, \mathrm{Re}(\alpha) > 0.$

It’s generalized form is given by Wiman [3]

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)} \tag{2}$$

where $\alpha, \beta \in C, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0.$

The generalization of the above functions is given by Prabhakar [23] in 1971 in the form

$$E_{\alpha,\beta}^{\gamma}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n x^n}{n!\Gamma(\alpha n + \beta)}$$

where $\alpha, \beta, \gamma \in C, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta), \mathrm{Re}(\gamma) > 0.$

(3)

Sharma [23] defined the M-series as

$$ {}_pM_q^{\alpha}(a_1,\ldots,a_p; b_1,\ldots,b_q; x) = {}_pM_q^{\alpha}(x) = \sum_{r=0}^{\infty} \frac{(a_1)_r \ldots (a_p)_r}{(b_1)_r \ldots (b_q)_r} \frac{x^r}{\Gamma(\alpha r + 1)} \tag{4}$$

where $\alpha \in C, \mathrm{Re}(\alpha) > 0$ and $(a_j)_r$ and $(b_j)_r$ are the Pochammer symbols. The detailed information of this series is given in [23].

A generalization of (5) was defined by Sharma et al. [15] as

$$ {}_pM_q^{\alpha,\beta}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = {}_pM_q^{\alpha,\beta}(x) = \sum_{r=0}^{\infty} \frac{(a_1)_r\ldots(a_p)_r}{(b_1)_r\ldots(b_q)_r} \frac{x^r}{\Gamma(\alpha r+\beta)} \tag{5} $$

where $\alpha,\beta \in C, \mathrm{Re}(\alpha) > 0$ and $(a_j)_r$ and $(b_j)_r$ are the Pochammer symbols. The detailed information of this series is given in [].

2. Main Results

In this section, we derive the recurrence relations of generalized M-series:

Result 1 ${}_rM_s^{\alpha,\beta}(x) = {}_rM_s^{\alpha,\beta+1}(x) + ax\frac{d}{dx}{}_rM_s^{\alpha,\beta+1}(x)$

$$ \tag{6} $$

Proof:
Using (6) on RHS we arrive at the desired result.

Result 2 $\left(\frac{d}{dx}\right)^m {}_rM_s^{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{r}(a_j)_n}{\prod_{j=1}^{s}(b_j)_n} \frac{(1+m)_n(n+1)_m}{\Gamma(\alpha n+\alpha m+\beta)(1+m)_n} x^n$

$$ \tag{7} $$

Proof:
Same as above.

Result 3 $\left(\frac{d}{dx}\right)^m x^{\beta-1}{}_rM_s^{\alpha,\beta}(wx^{\alpha}) = x^{\beta-m-1}{}_rM_s^{\alpha,\beta-m}(x)$

$$ \tag{8} $$

Proof:
Same as above.

Result 4 $\int_t^z (s-t)^{\beta-1}{}_rM_s^{\alpha,\beta}\left(\lambda(s-t)^{\alpha}\right)ds = (z-t)^{\beta-1}{}_rM_s^{\alpha,\beta}\left(\lambda(s-t)^{\alpha}\right)$

$$ \tag{9} $$

Proof:

Let $u = \frac{s-t}{z-t}$ we get

$$ \int_t^z (s-t)^{\beta-1}{}_rM_s^{\alpha,\beta}\left(\lambda(s-t)^{\alpha}\right)ds $$

$$ = \int_t^z (z-t)^{\beta-1}u^{\beta-1}(z-t) \times \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{r}(a_j)_n}{\prod_{j=1}^{s}(b_j)_n} \frac{\lambda^n(z-t)^{\alpha n}u^{n\alpha}(1+m)_n(n+1)_m}{\Gamma(\alpha n+\beta)} du \tag{10} $$

On simplifying we arrive at the desired result.
In particular, setting $t = 0$ and $z = 1$, we get

$$\int_0^1 u^{\beta-1}(1-u)^{\delta-1}\,{}_rM_s^{\alpha,\beta}(xu^{\alpha})ds = {}_rM_s^{\alpha,\beta+1}(x) \tag{11}$$

Remarks: If we take $r = s = 0$ in theorems (5.1) to (5.4), we obtain the results given earlier by Salim et al. [20].

Conflict of Interest

The authors declare that there is no conflict of interest related to this research work.

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