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The Generalized K-Function and Its Recurrence Relations

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Abstract

The aim of this paper is to investigate the recurrence relations of the generalized K- function introduced earlier by Sharma [10]. Several special cases have also been discussed.
Keywords: - Fractional calculus, K-function, Recurrence relation.

1. Introduction

Fractional Calculus deals with derivatives and integrals of arbitrary orders. During the last three decades, Fractional Calculus has been applied to almost every field of Mathematics such as Special Functions etc., Science, Engineering and Technology. Many applications of Fractional Calculus can be found in Turbulence and Fluid Dynamics, Stochastic Dynamical System, Plasma Physics and Controlled Thermonuclear Fusion, Non-linear Control Theory, Image Processing, Non-linear Biological Systems and Astrophysics.
The Mittag-Leffler function has gained importance and popularity during the last decade due mainly to its applications in the solution of fractional-order differential, integral and difference equations arising in certain problems of mathematical, physical, biological and engineering sciences.
In 1993, this function was introduced and studied by the Swedish Mathematics Gosta Mittag-Leffler [5, 6] in terms of the power series.

$$E_{\alpha}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\alpha n+1)},\quad (\alpha>0) \tag{1.1}$$

In 1905, a generalization of this series was introduced in the following form

$$E_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\alpha n+\beta)},\qquad (\alpha,\beta>0) \tag{1.2}$$

And studied by several authors notably MittagLeffler [5, 6], Wiman [2], Agrawal [13], Humbert and Agrawal [12] and Dzrbashjan[8, 9, 11]. It is shown in [14] that the function defined by (1.1) and (1.2) are both entire functions of order $\rho=1$ and type $\sigma=1$.

1971, Prabhakar [15] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ defined by

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \quad (\alpha,\beta,\gamma\in C \;\; \Re(\alpha)>0, \;\; \Re(\beta)>0, \;\; \Re(\gamma)>0, \tag{1.3}$$

Where $(\gamma)_n$ is the Pochammer symbol (see e.g..[3]) defined $(\lambda \in C)$ by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases} 1 & (n=0;\lambda\neq 0) \\ \lambda(\lambda+1)..........(\lambda-n+1) & (n\in N;\lambda\in C)\end{cases} \tag{1.4}$$

N being the set of positive integers

In 2012, G.A. Dorrego and R.A. Cerutti [7] introduced k-Mittag-Leffler function $E_{k,\alpha,\beta}^{\gamma}(z)$ defined by

$$E_{k,\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)}\frac{z^n}{n!} \tag{1.5}$$

Where $(\gamma)_{n,k}$ is the k-Pochammer symbol and $\Gamma_k(x)$ is the Gamma function as given in

$$(x)_{n,k}=x(x+k)(x+2k)(x+3k)........(x+(n-1)k), \quad \gamma\in C, k\in\Re \; and \; n\in N$$

and

$$\Gamma_k(z)=\int_0^{\infty} e^{-\frac{t^k}{k}}t^{z-1}dt, \;\; k\in\Re, z\in C$$

Particularly, $\Gamma_k(x)\to\Gamma x \; as \;\; k\to 1$

The generalization of (1.1) to (1.5) was defined by Sharma[9] as

$${}_pK_q^{\alpha,\beta;\gamma}(a_1,\dots,a_p;b_1,\dots,b_q;x)={}_pK_q^{\alpha,\beta;\gamma}(x)=\sum_{r=0}^{\infty}\frac{(a_1)_r\dots(a_p)_r}{(b_1)_r\dots(b_q)_r}\frac{(\gamma)_r x^r}{r!\Gamma(\alpha r+\beta)} \tag{1.6}$$

where $\alpha,\beta,\gamma\in C, \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(\gamma)>0$, $(a_j)_r$ and $(b_j)_r$ are the Pochammer symbols. The detailed information of this function is given in [9] and some new properties of it are recently obtained by Sharma [10].

In 2015, a new generalization of (1.6) was introduced by Sharma and defined as

$${}_rK_s^{\alpha,\beta;\gamma,\delta}(a_1,\dots,a_r;b_1,\dots,b_s;x)={}_rK_s^{\alpha,\beta;\gamma,\delta}(x)=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{r}(a_j)_n}{\prod_{j=1}^{s}(b_j)_n}\frac{(\gamma)_n x^n}{(\delta)_n\Gamma(\alpha n+\beta)} \tag{1.7}$$

Where

$$\alpha,\beta,\gamma,\delta\in C, \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(\gamma)>0, \mathrm{Re}(\delta)>0, \tag{1.8}$$

and $(a_j)_n$ and $(b_j)_n$ are the Pochammer symbols.

2. Recurrence Relation

In this section, we have derived the recurrence relation of the generalized K- function. The result is represented in the form of the following theorem:

Theorem: 1. If $\mathrm{R}(\alpha+p)>0,\quad \mathrm{R}(\beta+s)>0,\quad \mathrm{R}(\gamma)>0,\quad \mathrm{R}(\delta)>0,$

Then

$$ {}_rK_s^{\alpha+p,\beta+s+1;\gamma,\delta}(x) - {}_rK_s^{\alpha+p,\beta+s+2;\gamma,\delta}(x) = (\beta+s)(\beta+s+2)\,{}_rK_s^{\alpha+p,\beta+s+3;\gamma,\delta} $$

$$ +(\alpha+p)^2x^2\frac{d^2}{dx^2}\,{}_rK_s^{\alpha+p,\beta+s+3;\gamma,\delta}(x) + (\alpha+p)\{\alpha+p+2(\beta+s+1)\}x\frac{d}{dx}\,{}_rK_s^{\alpha+p,\beta+s+3;\gamma,\delta}(x) \tag{2.1} $$

It is easy to obtain the following corollary by letting $\alpha+p=v\ and\ \beta+s=m$

Corollary 1. We have, for $v,m\in N$

$$ {}_rK_s^{\upsilon,m+1;\gamma,\delta}(x) = {}_rK_s^{\upsilon,m+2;\gamma,\delta}(x) + m(m+2)\,{}_rK_s^{\upsilon,m+3;\gamma,\delta} $$

$$ +\upsilon^2x^2\frac{d^2}{dx^2}\,{}_rK_s^{\upsilon,m+3;\gamma,\delta}(x) + \upsilon\{\upsilon+2(m+1)\}x\frac{d}{dx}\,{}_rK_s^{\upsilon,m+3;\gamma,\delta}(x) \tag{2.2} $$

Proof of the theorem 1. By applying the fundamental relation of the Gamma function $\Gamma(z+1)=z\Gamma z$ to (1.7), we can write

$$ {}_rK_s^{\alpha+p,\beta+s+1;\gamma,\delta}(x) = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{r}(a_j)_n}{\prod_{j=1}^{s}(b_j)_n}\frac{(\gamma)_n x^n}{(\delta)_n\{(\alpha+p)n+\beta+s\}\Gamma_k((\alpha+p)n+\beta+s)} \tag{2.3} $$

and

$$ {}_rK_s^{\alpha+p,\beta+s+2;\gamma,\delta}(x) = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{r}(a_j)_n}{\prod_{j=1}^{s}(b_j)_n}\frac{(\gamma)_n x^n}{(\delta)_n\{(\alpha+p)n+\beta+s\}\Gamma_k((\alpha+p)n+\beta+s)} \tag{2.4} $$

Equation can be written as follows:

$$ {}_rK_s^{\alpha+p,\beta+s+2;\gamma,\delta}(x) = $$

$$ \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{r}(a_j)_n}{\prod_{j=1}^{s}(b_j)_n}\left\{\frac{1}{(\alpha+p)n+\beta+s}-\frac{1}{(\alpha+p)n+\beta+s+1}\right\}\frac{(\gamma)_n x^n}{(\delta)_n\Gamma_k((\alpha+p)n+\beta+s)} $$

$$ = {}_rK_s^{\alpha+p,\beta+s+1;\gamma,\delta}(x) - \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{r}(a_j)_n}{\prod_{j=1}^{s}(b_j)_n}\frac{(\gamma)_n x^n}{(\delta)_n\{(\alpha+p)n+\beta+s+1\}\Gamma_k((\alpha+p)n+\beta+s)} \tag{2.5} $$

We, for convenience, have denoted the last summation in (2.5) by S:

$$S = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{r} (a_j)_n}{\prod_{j=1}^{s} (b_j)_n} \frac{(\gamma)_n x^n}{\{(\alpha+p)n+\beta+s+1\} \Gamma_k ((\alpha+p)n+\beta+s)} \tag{2.6}$$

$${}_rK_s^{\alpha+p,\beta+s+1;\gamma,\delta}(x) - {}_rK_s^{\alpha+p,\beta+s+2;\gamma,\delta}(x)$$

Applying a simple identity

$$\frac{1}{u} = \frac{1}{u(u+1)} + \frac{1}{u+1} \quad where \ \{u = (\alpha+p)n+\beta+s+1\},$$

we obtain the desired result.

3. Conclusion

The study has established the various properties of the generalized K-Function and its recurrence relations. The presentations discussed in this study will make the reader familiar with the present trends of research in generalized K-Function and will help in fostering their applications.

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Conflict of Interest

The Authors declares that there is no potential conflict of interest in this manuscript.

References

[1] Shukla, A. K., and Prajapati, J. C. (2007). On a generalization of Mittag-Leffler function and its properties. J. Math. Anal., 336, 797-811.
[2] Wiman, A. (1905). Uber die Nullsteliun der Fuctionen E□(x). Acta Math. 29, 217-234.
[3]. Rainville, E.D. (1960). Special functions. Chelsea Publishing Company, Bronux, New York.
[4]. Gupta, I. S., and Debuath, L. (2007). Some properties of the Mittag-Leffler functions. integral Trans. Spec. Funct., 18, 329-336.
[5]. Mittag-Leffler, G. (1903). Sur la nuovelle function E(x). Comptes Rendus de l'Academie des Sciences Paris, 37, 554-558.
[6]. Mittag-Leffler, G. M. (1905). Sur la representation analytique de'une branche uniforme une function monogene. Acta. Math. 29, 101-181.
[7]. Dorrego, G.A., and Cerutti, R.A. (2012). The k-Mittag-Leffler Function. Int. J. Contemp. Math. Sciences, 7, 705-716.
[8]. Dzrbashjan, M. M. (1952). On the integral representation and uniqueness of some classes of entire functions (in Russian), Dokl. AN SSSR, 85, 29-32.
[9]. Dzrbashjan, M. M. (1960). On the integral transformations generated by the generalized Mittag-Leffler function (in Russian) Izv. AN Arm. SSR 13, 21-63.
[10]. Sharma, K. (2015). Some Properties of a Generalized Mittag-Leffler Function. J. Indian Math. Soc., 82, 155-167.
[11]. Dzrbashjan, M. M. (1966). Integral transforms and Representations of Functions in the complex Domain (in Russian) Nauka, Moscow.
[12]. Humbert, P., and Agrawal, R. P. (1953). Sur la function de Mittag-Leffler et quelquesunes de ses. Generalizations. Bull Sci. Math., 2, 180-185.
[13]. Agarwal, N. (1953). A Propos d'unc Note de H4. Pierre Humbert. C. R. Se'ances Acad. Sci., 236, 2031-2032.
[14]. Gorenflo, R., Kilbas, A. A., and Rogosin, S. V. (1997). On the generalized Mittag- Leffler type functions. Intehgral Transforms and Special Functions, 7, 215-224. https://doi.org/10.1080/10652469808819200
[15]. Prabhakar, T. R. (1971). A singular integral equation with a generalized Mittag-Leffler function in the Kernel. Yokohama Math. J., 19, 7-15.