International Journal of Innovative Research & Growth
A Peer Reviewed Journal
E-ISSN: 2455-1848
homepage: http://www.ijirg.com/ijirg/
Volume: 13 Issue: 1, January 2024

Fractional Calculus and Generalized K-Function

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Article Info
Received 27 November 2023
Received in revised form 28 December 2023
Accepted for publication 13 January 2024
DOI: 10.26671/IJIRG.2024.1.13.101
Cited as
Bajpai, M., Sharma, K., and Saxena, V. (2024). Fractional Calculus and Generalized K-Function. *Int J Innovat Res Growth*, 13, 8-13.

Abstract

The present paper deals with the representation of the generalized K-function, which is an extension of the multi-index Mittag-Leffler function defined by Kiryakova [9], the topic has been introduced and studied by the author in terms of some special functions. it investigates the relations that exists between the generalized K-function and the operators of Riemann-Liouville fractional integrals and derivatives.
Mathematics Subject Classification: 26A33,33C60
Keywords: - Fractional calculus, Riemann- Liouville fractional integrals and derivatives.

1. Introduction

The Mittag-Leffler function has gained importance and popularity during the last decade mainly due to its applications in the solution of fractional-order differential, integral and difference equations arising in certain problems of mathematical, physical, biological and engineering sciences. This function is introduced and studied by Mittag-Leffler [10,11] in terms of the power series.

$$E_{\alpha}(z)=\sum_{r=0}^{\infty}\frac{z^{r}}{\Gamma(\alpha r+1)},\quad (\alpha>0) \tag{1.1}$$

A generalization of this series comes in the following form

$$E_{\alpha,\beta}(z)=\sum_{r=0}^{\infty}\frac{z^{r}}{\Gamma(\alpha r+\beta)},\quad (\alpha,\beta>0) \tag{1.2}$$

which has been studied by several authors notably Mittag-Leffler [10,11], Wiman [13], Agrawal [15], Humbert and Agrawal [8] and Dzrbashjan [1,2,3]. It is shown in [5] that the function defined by (1.1) and (1.2) are both entire functions of order $\rho=1$ and type $\sigma=1$. A detailed account of the basic properties of these two functions are given in the third volume of Bateman manuscript project[4] and an account of their various properties can be found in [2,12].
An interesting generalization of (1.2) is recently introduced by Kilbas and Saigo[16] in terms of a special entire function of the form

$$E_{\alpha,m,l}(z)=\sum_{r=0}^{\infty}c_{r}\,z^{r}, \tag{1.3}$$

where $c_r = \prod_{i=0}^{r-1} \frac{\Gamma[\alpha(im+l)+1]}{\Gamma[\alpha(im+l+1)+1]}$, $(n = 0,1,2,....)$

and an empty product is to be interpreted as unity. Certain properties of this function are associated with fractional integrals and derivatives [12].
The multiindex Mittag-Leffler function is defined by Kiryakova[9] by means of the power series

$$E_{(\frac{1}{\rho_i}),(\mu_i)}(z) = \sum_{r=0}^{\infty} \varphi_r z^r = \sum_{r=0}^{\infty} \frac{z^r}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{r}{\rho_j})}$$

where $m > 1$ is an integer, ρ_j and μ_j are arbitrary real numbers.

The multi index Mittag-Leffler function is an entire function that gives its asymptotic, estimate, order and type see Kiryakova[9].
The Wright generalized hypergeometric function [17] is given by

$$ {}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{matrix}(a_1,A_1),...,(a_p,A_p)\\(b_1,B_1),...,(b_q,B_q)\end{matrix};z\right] = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(a_i + rA_j)\, z^r}{\prod_{j=1}^{q}\Gamma(b_i + rB_j) r!} \tag{1.4}$$

[$A_j > 0(j = 1,2,..., p), B_j > 0(j = 1,2,..., q); 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0$ (Equality only for approximately bounded z)].
It is provided that the Riemann-Liouville fractional integral and derivative of the Wright function is also the Wright function but of greater order. Conditions for the existence of the series (1.4) together with its presentation in terms of the Mellin-Barnes integral and of the H-function were established in [18].
When $A_1 = ... = A_p = B_1 = ... = B_q = 1$, (1) reduces to ${}_pF_q(.)$:

$$ {}_p\Psi_q\left[\begin{matrix}(a_1,1),...,(a_p,1)\\(b_1,1),...,(b_q,1)\end{matrix};z\right] = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)} {}_pF_q\,(a_1,..., a_p; b_1,..., B_q; z) \tag{1.5}$$

where $p \leq q, |z| < \infty; p = q+1; |z| < 1; p = q+1; |z| = 1, \mathrm{Re}(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j) > 0.$

The present paper is organized as follows: In section 2, we give the definition of the Generalized K-function and its relation with another special functions, namely multi-index Mittag-Leffler function, Mittag-Leffler function, generalized Mittag-Leffler function and exponential function. In section 3, certain relations that exists between generalized K-function and the operators of Riemann-Liouville fractional calculus are investigated.

2. The Generalized K-Function

The generalized K-function introduced by the author is defined as follows:

$$ {}_pK_q^{\frac{1}{\rho_j},\mu_j}(a_1,..., a_p; b_1,..., b_q; z) = {}_pK_q^{\frac{1}{\rho_j},\mu_j}(z) = \sum_{r=0}^{\infty} \frac{(a_1)_r...(a_p)_r}{(b_1)_r...(b_q)_r} \frac{z^r}{\prod_{j=1}^{m}\Gamma(\mu_j + \frac{r}{\rho_j})} \tag{2.1}$$

where $m > 1$ is an integer, $\rho_j(> 0)$ and μ_j are arbitrary real numbers ,$(a_j)_r$ and $(b_j)_r$ are the Pochhammer symbols.

The series (2.1) is defined when none of the parameters $b_j s$, $j = 1,2,...,q$, is a negative integer or zero. If any numerator parameter a_{jr} is a negative integer or zero, the series terminates to a polynomial in z. From the ratio test it is evident that the series is convergent for all z if p > q + 1. When p = q + 1 and $|z| = 1$, the series can converge in some cases. Let $\gamma = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j$. It can be stated that when p = q + 1 the series is absolutely convergent for $|z| = 1$ if ($R(\gamma) < 0$, conditionally convergent for z = -1 if $0 \leq R(\gamma) < 1$ and divergent for $|z| = 1$ if $1 \leq R(\gamma)$.

2.1. Some Special Cases of Generalized K-Function are Given Below

(i) When there are no upper and lower parameters, we get

$$ {}_0K_0^{\frac{1}{\rho_j},\mu_j}(-;-;z) = \sum_{r=0}^{\infty} \frac{z^r}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{r}{\rho_j})} \tag{2.2} $$

which reduces to the multi-index Mittag-Leffler function defined by Kiryakova[9].

(ii) If we put $m = 2, p = q = 0$ in (2.1), we get

$$ {}_0K_0^{(\frac{1}{\rho_1},\frac{1}{\rho_1}),(\mu_1,\mu_2)}(-;-;z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\mu_1 + \frac{r}{\rho_1})\Gamma(\mu_2 + \frac{r}{\rho_2})} \tag{2.3} $$

It is shown by Dzrbashjan[3] that (2.2) is an entire function of order

$\rho = \frac{\rho_1 \rho_2}{\rho_1 + \rho_2}$ and type $\sigma = \left(\frac{\rho_1}{\rho}\right)^{\frac{\rho}{\rho_1}} \left(\frac{\rho_2}{\rho}\right)^{\frac{\rho}{\rho_2}}$

(iii) If we put $m = 1, \rho_j = \frac{1}{\alpha}, \mu_j = \beta, p = q = 0$ in (2.1), we get

$$ {}_0K_0^{\alpha,\beta}(-;-;z) = {}_1K_0^{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + \beta)} \tag{2.4} $$

which is the generalized Mittag-Leffler function denoted by $E_{\alpha,\beta}(z)$ see [10].

(iv) If we put $m = 1, \rho_j = \frac{1}{\alpha}, \mu_j = 1, p = q = 0$ in (2.1), we get the Mittag-Leffler function [10] denoted by $E_\alpha(z)$ Therefore

$$ E_\alpha(z) = {}_0K_0^{\alpha,1}(-;-;z) = {}_1K_0^{\alpha,1}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + 1)} \tag{2.5} $$

(v) If we take $\alpha = 1$ in (2.4), we get the Exponential function [14] denoted by e^x

Therefore

$$ E_\alpha(z) = e^x = \sum_{r=0}^{\infty} \frac{z^r}{r!} \tag{2.6} $$

3. Relations with Riemann-Liouville Fractional Calculus

In this section, we derive certain relations between generalized K-function and Riemann-Liouville Fractional Calculus.

Theorem 3.1 Let $\alpha > 0, \rho_j > 0, \mu_j > 0 (j = 1,2,...,m)$ and I_Z^α be the operator of Riemann-Liouville fractional integral, then there holds the relation:

$$I_Z^\alpha\, {}_pK_q^{\frac{1}{\rho_j},\mu_j}(z) = \frac{z^\alpha}{\Gamma(\alpha+1)}\; {}_{p+1}K_{q+1}^{\frac{1}{\rho_j},\mu_j}(a_1,\ldots,a_p,1; b_1,\ldots, b_q, \alpha+1; z) \tag{3.1}$$

Proof: Following Section 2 of the book by Samko, Kilbas and Marichev [8], the fractional Riemann-Liouville(R-L) integral operator (For lower limit a=0 w. r. t. variable z) is given by

$$I_Z^\alpha f(x)\,(z) = \frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1} f(t)dt \tag{3.2}$$

By virtue of (3.2) and (2.1), we obtain

$$I_Z^\alpha\, {}_pK_q^{\frac{1}{\rho_j},\mu_j}(z) = \frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}\sum_{r=0}^{\infty}\frac{(a_1)_r\cdots(a_p)_r}{(b_1)_r\cdots(b_q)_r}\frac{t^r}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{r}{\rho_j})}dt \tag{3.3}$$

Interchanging the order of integration and evaluating the inner integral with the help of Beta function, it gives

$$I_Z^\alpha\, {}_pK_q^{\frac{1}{\rho_j},\mu_j}(z) = \frac{z^\alpha}{\Gamma(\alpha+1)}\sum_{r=0}^{\infty}\frac{(a_1)_r\cdots(a_p)_r(1)_r z^r}{(b_1)_r\cdots(b_q)_r(\alpha+1)_r\prod_{j=1}^{m}\Gamma(\mu_j+\frac{r}{\rho_j})}$$

$$= \frac{z^\alpha}{\Gamma(\alpha+1)}\; {}_{p+1}K_{q+1}^{\frac{1}{\rho_j},\mu_j}(a_1,\ldots,a_p,1; b_1,\ldots, b_q, \alpha+1; z) \tag{3.4}$$

The interchange of the order of integration and summation is permissible under the conditions stated along with the theorem due to convergence of the integrals involved in this process.
That is, as naturally expected for fractional calculus operators of special functions being generalized hypergeometric functions, a Riemann-Liouville fractional integral of the K-function is again the K-function with the indices p+1, q+1. This completes the proof of the theorem (3.1).

Theorem 3.2 Let $\alpha > 0, \rho_j > 0, \mu_j > 0 (j = 1,2,...,m)$ and D_Z^α be the operators of Riemann-Liouville fractional derivative, then there holds the relation

$$D_Z^\alpha\, {}_pK_q^{\frac{1}{\rho_j},\mu_j}(z) = \frac{z^{-\alpha}}{\Gamma(1-\alpha)}\; {}_{p+1}K_{q+1}^{\frac{1}{\rho_j},\mu_j}(a_1,\ldots,a_p,1; b_1,\ldots, b_q, 1-\alpha; z) \tag{3.5}$$

Proof: Following Section 2 of the book by Samko, Kilbas and Marichev[8], the fractional Riemann-Liouville(R-L) integral operator(For lower limit a = 0 w. r. t. variable z) is given by

$$D_Z^\alpha f(z) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dz}\right)^n\int_0^z (z-t)^{n-\alpha-1} f(t)dt \tag{3.6}$$

where $n = [\alpha]+1$.

From (2.1) and (3.6), it follows that

$$D_{z}^{\alpha}\ {}_pK_q^{\frac{1}{\rho_j},\mu_j}(z)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dz})^n\int_0^z (z-t)^{n-\alpha-1}\sum_{r=0}^{\infty}\frac{(a_1)_r\cdots(a_p)_r}{(b_1)_r\cdots(b_q)_r}\frac{t^r}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{r}{\rho_j})}dt \tag{3.7}$$

Interchanging the order of integration and evaluating the inner integral with the help of Beta function, it gives

$$D_{z}^{\alpha}\ {}_pK_q^{\frac{1}{\rho_j},\mu_j}(z)=\frac{z^{\alpha}}{\Gamma(1-\alpha)}\sum_{r=0}^{\infty}\frac{(a_1)_r\cdots(a_p)_r(1)_r}{(b_1)_r\cdots(b_q)_r(1-\alpha)_r}\frac{t^r}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{r}{\rho_j})}$$

$$=\frac{z^{-\alpha}}{\Gamma(1-\alpha)}\ {}_{p+1}K_{q+1}^{\frac{1}{\rho_j},\mu}(a_1,\ldots,a_p,1;b_1,\ldots,b_q,1-\alpha;z) \tag{3.8}$$

This shows that a Riemann-Liouville fractional integral of the K-function is again the K-function with the indices p+1, q+1.
This completes the proof of the theorem (3.2).

4. Conclusion

The study has established the various properties of the generalized K-Function and its recurrence relations. The presentations discussed in this study will make the reader familiar with the present trends of research in generalized K-Function and will help in fostering their applications.

Acknowledgements

The authors are thankful to the referees for giving useful suggestions in the improvement of the paper.

Conflict of Interest

The Authors declares that there is no potential conflict of interest in this manuscript.

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