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Some aspects of the group invariant solutions of wave propagation for an electric field

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Abstract

This paper deals with the investigation of the solution of wave propagation of an electric field in a source free, linear, isotropic, homogeneous region described by the Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

where u is wave function, k is wave number, ∇ is Laplacian-operator. The solution obtained from the general prolongation formula for the closed form of the symmetry group. The results presented here are general in nature and include the results previously examined by several authors. The benefit of using the general prolongation method is that the invariant solution of the Helmholtz equation involves linearity, rotation and scaling symmetries.
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1. Introduction

The Helmholtz equation arises naturally in many physical applications related to wave propagation, vibrational phenomena, heat transfer, optics, acoustics, electrostatics and quantum mechanics. These equations are commonly used to describe structural vibrations, acoustic cavity problems, radiated waves, scattering of waves, heat conduction in fins and acoustic scattering in fluid solids. This Helmholtz equation is a general equation that can be found in many areas of physics. The Helmholtz equation is a form of linear or nonlinear partial differential equation. Bao et al., studied the discrete singular convolution algorithm for solving the Helmholtz equation with high wave-numbers (Bao et al., 2004). The algebraic multilevel preconditioning for the Helmholtz equation with high wave numbers investigated in (Bollhöfer et al., 2009) and (Erlangga et al., 2004). Shiralashetti et al., derived numerical solution of Helmholtz equation using modified wavelet multigrid method (Shiralashetti et al., 2020). Li et al., proposed a deep-learning-based Robin-Robin domain decomposition method for Helmholtz equations (Li et al., 2023). Hirtum Obtained the solution of two-dimensional Helmholtz equation in the physical domain by applying the conformal map (Hirtum, 2017). Sakkaravarthi et al., determined the Lie point symmetries of nonlinear Helmholtz equation (Sakkaravarthi et al., 2018). El-Tantawy et al., derived the analytical and approximate solutions to the damped quadratic nonlinear Helmholtz equation using Runge-Kutta fourth-order method, finite difference method and homotopy perturbation method (El-Tantawy et al., 2021). Gupta & Pal obtained the most general solution of harmonic wave excitation in a semi-infinite medium (Gupta & Pal, 2013). Pal et al., investigated group analysis for Klein-Gordon equation via their symmetries (Pal et al., 2023) and also studied the application of lie group in two-dimensional heat equation (Pal et al., 2022). Shekhawat et al., obtained the symmetry group solution of wave propagation for vibrating uniform membrane by using the general prolongation formula (Shekhawat et al., 2022). The governing wave equation of an electric field E in a source free, linear, isotropic, homogeneous region described by Helmholtz equation

$$\nabla^2 u + \mathrm{k}^2 u = 0 \tag{1}$$

where u is wave function, k is wave number, ∇ is Laplacian operator (Pozar, 2011).

2. Methodology

2.1 The General Prolongation Formula

Let $$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{2}$$

be a vector field defined on an open subset $M \subset X \times U$ where X is the space of independent variables, and U is the space of dependent variables, p is the number of independent variables and q is the number of dependent variables for the system. Then n^{th}-prolongation of v is defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$ where X is the space of the independent variables, $U^{(n)}$ is the space of the dependent variables and the derivative of the dependent variables up-to n (order of differential equation) by

$$pr^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q}\sum_{J} \varphi_\alpha^J\left(x,u^{(n)}\right)\frac{\partial}{\partial u_J^\alpha} \tag{3}$$

The second summation being over all unordered multi-indices $J = (j_1, j_2, \ldots, j_k)$ with $1 \le j_k \le p$, $1 \le k \le n$. The coefficient function ϕ_α^{J} of $pr^{(n)}\mathbf{v}$ are given by the formula

$$\phi_\alpha^{\mathrm{J}}\left(x,u^{(n)}\right) = D_J\left(\phi_\alpha - \sum_{i=1}^{p}\xi^i u_i^\alpha\right) + \sum_{i=1}^{p}\xi^i u_{J,i}^\alpha \tag{4}$$

where $u_i^\alpha = \left(\partial u^\alpha / \partial x^i\right)$ and $u_{J,i}^\alpha = \left(\partial u_J^\alpha / \partial x^i\right)$(Olver, 1993).

2.2 Theorem

Suppose $\Delta_d(x, u^{(n)}) = 0$, for $d = 1,\ldots,l$ is a system of differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M, and

$$pr^{(n)}\mathbf{v}\,[\Delta_d(x, u^{(n)})] = 0 \quad (\text{for } d = 1,\ldots,l, \text{ whenever } \Delta(x, u^{(n)}) = 0) \tag{5}$$

for every infinitesimal generator $\mathbf{v}$ of G, then G is a symmetry group of the system. We can replace (5) by the equivalent condition that there exists function $Q_{d\mu}(x, u^{(n)})$ (where $Q_{d\mu}: M \to R$, for all μ, $d = 1,\ldots,l$) such that

$$pr^{(n)}\mathbf{v}\left[\Delta_d\left(x,u^{(n)}\right)\right] = \sum_{\mu=1}^{l} Q_{d\mu}\left(x,u^{(n)}\right)\Delta_\mu\left(x,u^{(n)}\right) \tag{6}$$

holds identically in $(x, u^{(n)}) \in M^{(n)}$ (Olver, 1993).

3. Result

3.1 Solution of the Helmholtz Equation

The solution of Helmholtz equation

$$\nabla^2 u + \mathrm{k}^2 u = 0 \qquad (7)$$

for wave propagation of an electric field E in a source free, linear, isotropic, homogeneous region, is the second order differential equation with three independent variables and one dependent variable (notation given in section 2.1 are as p = 3, n = 2 and q = 1). Now, let

$$\boldsymbol{v} = \xi\,\partial_x + \eta\,\partial_y + \zeta\,\partial_z + \phi\,\partial_u \qquad (8)$$

be a vector field on X×U. Then the smooth coefficient functions $\xi = \xi(x, y, z)$, $\eta = \eta(x, y, z)$, $\zeta = \zeta(x, y, z)$ and ϕ are determined so that the corresponding one parameter groups exp(ε***v***) is a symmetry group of the Helmholtz equation. By theorem (2.2) we determine the second prolongation

$$pr^{(2)}v = v + \phi^x\partial_{u_x} + \phi^y\partial_{u_y} + \phi^z\partial_{u_z} + \phi^{xx}\partial_{u_{xx}} + \phi^{xy}\partial_{u_{xy}} + \phi^{xz}\partial_{u_{xz}} + \phi^{yy}\partial_{u_{yy}} + \phi^{yz}\partial_{u_{yz}} + \phi^{zz}\partial_{u_{zz}} \qquad (9)$$

of ***v***, the coefficients present in equation (9) can be calculated by using equation (4). Applying $pr^{(2)}\boldsymbol{v}$ to equation (7), the infinitesimal criterion equation (6) takes the form

$$\phi^{xx} + \phi^{yy} + \phi^{zz} + \mathrm{k}^2\phi = Q\left(u_{xx} + u_{yy} + u_{zz} + \mathrm{k}^2 u\right) \qquad (10)$$

in which $Q(x, y, z, u^{(2)})$. By substituting the values of ϕ^{xx}, ϕ^{yy} and ϕ^{zz} in equation (10) and equating the coefficients of the terms in the first and second order partial derivatives of *u*, the determining equations for symmetry group of the Helmholtz equation are found as follows

Monomial	Coefficients	Equation Number
$u_y\,u_{zy}$	$-2\zeta_u = 0$	(11)
$u_x\,u_{zx}$	$-2\zeta_u = 0$	(12)
$u_z\,u_{yz}$	$-2\eta_u = 0$	(13)
$u_x\,u_{yx}$	$-2\eta_u = 0$	(14)
$u_z\,u_{xz}$	$-2\xi_u = 0$	(15)
$u_y\,u_{xy}$	$-2\xi_u = 0$	(16)
$u_z\,u_{zz}$	$-3\zeta_u = 0$	(17)
$u_z\,u_{yy}$	$-\zeta_u = 0$	(18)
$u_z\,u_{xx}$	$-\zeta_u = 0$	(19)
$u_y\,u_{zz}$	$-\eta_u = 0$	(20)
$u_y\,u_{yy}$	$-3\eta_u = 0$	(21)
$u_y\,u_{xx}$	$-\eta_u = 0$	(22)
$u_x\,u_{zz}$	$-\xi_u = 0$	(23)
$u_x\,u_{yy}$	$-\xi_u = 0$	(24)
$u_x\,u_{xx}$	$-3\xi_u = 0$	(25)
u_{zz}	$-\phi_u + 2\zeta_z = Q$	(26)
u_{yy}	$-\phi_u + 2\eta_y = Q$	(27)
u_{xx}	$-\phi_u + 2\xi_x = Q$	(28)
u_{zx}	$-2\zeta_x - 2\xi_z = 0$	(29)

u_{yz}	$-2\zeta_y - 2\eta_z = 0$	(30)
u_{xy}	$-2\eta_x - 2\xi_y = 0$	(31)
$u_z u_y^2$	$-\zeta_{uu} = 0$	(32)
$u_z u_x^2$	$-\zeta_{uu} = 0$	(33)
$u_y u_z^2$	$-\eta_{uu} = 0$	(34)
$u_y u_x^2$	$-\eta_{uu} = 0$	(35)
$u_x u_z^2$	$-\xi_{uu} = 0$	(36)
$u_x u_y^2$	$-\xi_{uu} = 0$	(37)
$u_z u_x$	$-2\zeta_{xu} - 2\xi_{zu} = 0$	(38)
$u_y u_z$	$-2\zeta_{yu} - 2\eta_{zu} = 0$	(39)
$u_x u_y$	$-2\eta_{xu} - 2\xi_{yu} = 0$	(40)
u_z^3	$-\zeta_{uu} = 0$	(41)
u_y^3	$-\eta_{uu} = 0$	(42)
u_x^3	$-\xi_{uu} = 0$	(43)
u_z^2	$\phi_{uu} - 2\zeta_{zu} = 0$	(44)
u_y^2	$\phi_{uu} - 2\eta_{yu} = 0$	(45)
u_x^2	$\phi_{uu} - 2\xi_{xu} = 0$	(46)
u_z	$2\phi_{zu} - \zeta_{xx} - \zeta_{yy} - \zeta_{zz} = 0$	(47)
u_y	$2\phi_{yu} - \eta_{xx} - \eta_{yy} - \eta_{zz} = 0$	(48)
u_x	$2\phi_{xu} - \xi_{xx} - \xi_{yy} - \xi_{zz} = 0$	(49)
1	$\mathrm{k}^2\phi + \phi_{xx} + \phi_{yy} + \phi_{zz} - \mathrm{k}^2 u\, Q = 0$	(50)

The requirement for the equation (11) to (25) is that ξ, η, ζ are independent of u and from (44) or (45) or (46) we obtained $\phi = \beta(x, y, z)\, u + \alpha(x, y, z)$ where $\alpha(x, y, z)$ and $\beta(x, y, z)$ are functions. The equation (29) to (31) gives $\eta_x = -\xi_y$, $\zeta_y = -\eta_z$, $\zeta_x = -\xi_z$ and the equation (26) to (28) yield the relations $\xi_x = \eta_y = \zeta_z$. The coefficient of the equation from (47) to (49) gives $\beta_x = 0$, $\beta_y = 0$, $\beta_z = 0$. From the equation (11) we found $\beta = Q = \left(c_1/\mathrm{k}^2\right)$. Hence the most general infinitesimal symmetry of the Helmholtz equation has coefficient functions of the form

$\xi = c_7\, z + c_5\, y + c_2$,

$\eta = c_6\, z - c_5\, x + c_3$,

$\zeta = c_4 - c_7\, x - c_6\, y$, and

$\phi = \left(c_1/\mathrm{k}^2\right) u + \alpha$

where $c_1,\ldots, c_7$ are arbitrary constant and $\alpha(x, y, z)$ is an arbitrary solution of the Helmholtz equation for the wave propagation of an electric field E.

3.2 Lie Algebra of Infinitesimal Symmetries

The Lie algebra of infinitesimal symmetries of Helmholtz equation is spanned by the seven vector fields

$$\boldsymbol{v}_1 = \frac{u}{\mathrm{k}^2}\partial_u, \qquad \boldsymbol{v}_2 = \partial_x,$$

$$\boldsymbol{v}_3 = \partial_y, \qquad \boldsymbol{v}_4 = \partial_z,$$

$$\boldsymbol{v}_5 = y\partial_x - x\partial_y, \qquad \boldsymbol{v}_6 = z\partial_y - y\partial_z,$$

$$\boldsymbol{v}_7 = z\partial_x - x\partial_z \tag{51}$$

and the infinite-dimensional sub-algebra $\boldsymbol{v}_\alpha = \alpha(x, y, z)\partial_u$ where $\alpha(x, y, z)$ is an arbitrary solution of the Helmholtz equation for wave propagation of an electric field E.

The commutation relation between the vector fields is given by the following table, the entry in i^{th} row and j^{th} column represent $[\boldsymbol{v}_i, \boldsymbol{v}_j]$

	$\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4$	$\boldsymbol{v}_5$	$\boldsymbol{v}_6$	$\boldsymbol{v}_7$	$\boldsymbol{v}_\alpha$
$\boldsymbol{v}_1$	0	0	0	0	0	0	0	$-\left(\boldsymbol{v}_\alpha/\mathrm{k}^2\right)$
$\boldsymbol{v}_2$	0	0	0	0	$-\boldsymbol{v}_3$	0	$-\boldsymbol{v}_4$	$\boldsymbol{v}_{\alpha_x}$
$\boldsymbol{v}_3$	0	0	0	0	$\boldsymbol{v}_2$	$-\boldsymbol{v}_4$	0	$\boldsymbol{v}_{\alpha_y}$
$\boldsymbol{v}_4$	0	0	0	0	0	$\boldsymbol{v}_3$	$\boldsymbol{v}_2$	$\boldsymbol{v}_{\alpha_z}$
$\boldsymbol{v}_5$	0	$\boldsymbol{v}_3$	$-\boldsymbol{v}_2$	0	0	$-\boldsymbol{v}_7$	$\boldsymbol{v}_6$	$\boldsymbol{v}_{\alpha_1}$
$\boldsymbol{v}_6$	0	0	$\boldsymbol{v}_4$	$-\boldsymbol{v}_3$	$\boldsymbol{v}_7$	0	$-\boldsymbol{v}_5$	$\boldsymbol{v}_{\alpha_2}$
$\boldsymbol{v}_7$	0	$\boldsymbol{v}_4$	0	$-\boldsymbol{v}_2$	$-\boldsymbol{v}_6$	$\boldsymbol{v}_5$	0	$\boldsymbol{v}_{\alpha_3}$
$\boldsymbol{v}_\alpha$	$\left(\boldsymbol{v}_\alpha/\mathrm{k}^2\right)$	$-\boldsymbol{v}_{\alpha_x}$	$-\boldsymbol{v}_{\alpha_y}$	$-\boldsymbol{v}_{\alpha_z}$	$-\boldsymbol{v}_{\alpha_1}$	$-\boldsymbol{v}_{\alpha_2}$	$-\boldsymbol{v}_{\alpha_3}$	0

where $\alpha_1 = y\ \alpha_x - x\ \alpha_y$, $\alpha_2 = z\ \alpha_y - y\ \alpha_z$, $\alpha_3 = z\ \alpha_x - x\ \alpha_z$. The result assures that the totality of infinitesimal symmetries must be a Lie algebra (Olver, 1993), we conclude that if $\alpha(x, y, z)$ is any solution of the Helmholtz equation, so α_x, α_y, α_z, α_1, α_2, and α_3 is given above.

3.3 Symmetry Groups of The Helmholtz Equation

The one-parameter groups G_i (i = 1, …, 7, α) generated by the $\boldsymbol{v}_i$ are given by using $\exp(\varepsilon v_i)(x, y, z, u) = \left(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}\right)$ as follows

G_1: $(x, y, z, e^{(\varepsilon/k^2)} u)$, G_2: $(x + \varepsilon, y, z, u)$,
G_3: $(x, y + \varepsilon, z, u)$, G_4: $(x, y, z + \varepsilon, u)$,
G_5: $(x\cos\varepsilon + y\sin\varepsilon, y\cos\varepsilon - x\sin\varepsilon, z, u)$, G_6: $(x, y\cos\varepsilon + z\sin\varepsilon, z\cos\varepsilon - y\sin\varepsilon, u)$,
G_7: $(x\cos\varepsilon + z\sin\varepsilon, y, z\cos\varepsilon - x\sin\varepsilon, u)$, G_α: $(x, y, z, u + \varepsilon\alpha)$. (52)

Since each group G_i is a symmetry group (Olver, 1979).

3.4 Group Invariant Solutions of The Helmholtz Equation

The solution of Helmholtz equation corresponding to its different symmetry groups G_i (i =1,…, 7, α) are obtained by using $\tilde{u} = g \cdot u = g \cdot f(x, y, z)$ as follows

$u^{(1)} = e^{(\varepsilon/k^2)} f(x, y, z)$, $u^{(2)} = f(x - \varepsilon, y, z)$,
$u^{(3)} = f(x, y - \varepsilon, z)$, $u^{(4)} = f(x, y, z - \varepsilon)$,
$u^{(5)} = f(x\cos\varepsilon - y\sin\varepsilon, y\cos\varepsilon + x\sin\varepsilon, z)$,
$u^{(6)} = f(x, y\cos\varepsilon - z\sin\varepsilon, z\cos\varepsilon + y\sin\varepsilon)$,
$u^{(7)} = f(x\cos\varepsilon - z\sin\varepsilon, y, z\cos\varepsilon + x\sin\varepsilon)$,
$u^{(\alpha)} = f(x, y, z) + \varepsilon\alpha(x, y, z)$

where $u = f(x, y, z)$ be any given solution of the Helmholtz equation, ε is any real number and $\alpha(x, y, z)$ any other solution to the Helmholtz equation.

3.5 Most General Solution of The Helmholtz Equation

The most general solution of the Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

obtainable from a given solution $u = f(x, y, z)$ by using the groups of transformations $u = g\, f(x, y, z)$ representing as the composition of transformations in the various one parameter subgroups G_i (i = 1,…, 7, α), where $g = \prod_{i=1}^{7 \text{ and } \alpha} \exp(v_i)$ is an arbitrary group of transformation, in the form

$$u = e^{(\varepsilon_1/k^2)}\, f\big(x\cos\varepsilon_5.\cos\varepsilon_7 - y\sin\varepsilon_5 - z\sin\varepsilon_7 - \varepsilon_2,$$
$$y\cos\varepsilon_5.\cos\varepsilon_6 + x\sin\varepsilon_5 - z\sin\varepsilon_6 - \varepsilon_3,$$
$$z\cos\varepsilon_6.\cos\varepsilon_7 + y\sin\varepsilon_6 + x\sin\varepsilon_7 - \varepsilon_4\big) + \alpha(x, y, z) \quad (53)$$

where $\varepsilon_1, \dots, \varepsilon_7$ are real constant and α be an arbitrary solution of Helmholtz equation for wave propagation of an electric field E in a source free, linear, isotropic, homogeneous region.

4. Discussion

4.1 If we take k = 1, the equation (53) reduces to

$$u = e^{\varepsilon_1}\, f\big(x\cos\varepsilon_5.\cos\varepsilon_7 - y\sin\varepsilon_5 - z\sin\varepsilon_7 - \varepsilon_2,$$
$$y\cos\varepsilon_5.\cos\varepsilon_6 + x\sin\varepsilon_5 - z\sin\varepsilon_6 - \varepsilon_3,$$
$$z\cos\varepsilon_6.\cos\varepsilon_7 + y\sin\varepsilon_6 + x\sin\varepsilon_7 - \varepsilon_4\big) + \alpha(x, y, z) \quad (54)$$

where $\varepsilon_1, \dots, \varepsilon_7$ are real constant and α be an arbitrary solution of Helmholtz equation for wave propagation of an electric field E in a source free, linear, isotropic, homogeneous region.

4.2.1 If we take y = 0, the equation (7) reduces to the equation

$$u_{xx} + u_{zz} + k^2 u = 0 \quad (55)$$

and its solution is obtained from (53) as follows

$$u = e^{(\varepsilon_1/k^2)} f\left(x\cos\varepsilon_7 - z\sin\varepsilon_7 - \varepsilon_2,\ z\cos\varepsilon_7 + x\sin\varepsilon_7 - \varepsilon_4\right) + \alpha(x, z) \quad (56)$$

where $\varepsilon_1, \ldots, \varepsilon_7$ are real constant and α be an arbitrary solution of Helmholtz equation for wave propagation of an electric field E in a source free, linear, isotropic, homogeneous region (Shekhawat et al., 2022).

4.2.2 If we take y = 0 and k = 1, the equation (7) reduces to

$$u_{xx} + u_{zz} + u = 0 \quad (57)$$

and its solution is obtained from (56) as follows

$$u = e^{\varepsilon_1} f\left(x\cos\varepsilon_7 - z\sin\varepsilon_7 - \varepsilon_2,\ z\cos\varepsilon_7 + x\sin\varepsilon_7 - \varepsilon_4\right) + \alpha(x, z) \quad (58)$$

where $\varepsilon_1, \ldots, \varepsilon_7$ are real constant and α be an arbitrary solution of Helmholtz equation for wave propagation of an electric field E in a source free, linear, isotropic, homogeneous region (Shekhawat et al., 2022).

4.3.1 If we take y = 0 and z = 0, the equation (7) reduces to

$$u_{xx} + k^2 u = 0 \quad (59)$$

and its solution is obtained from (56) as follows

$$u = e^{(\varepsilon_1/k^2)} f(x-\varepsilon_2) + \alpha(x) \quad (60)$$

where ε_1, ε_2 are real constant and α be an arbitrary solution of Helmholtz equation for wave propagation of an electric field E in a source free, linear, isotropic, homogeneous region (Gupta & Pal, 2013).

4.3.2 If we take y = 0, z = 0 and k = 1, the equation (7) reduces to

$$u_{xx} + u = 0 \quad (61)$$

and its solution is obtained from (60) as follows

$$u = e^{\varepsilon_1} f(x-\varepsilon_2) + \alpha(x) \quad (62)$$

where ε_1, ε_2 are real constant and α be an arbitrary solution of Helmholtz equation for wave propagation of an electric field E in a source free, linear, isotropic, homogeneous region (Gupta & Pal, 2013).

5. Conclusion

The symmetry group provides a means of classifying different symmetry classes of solutions, where two solutions are deemed to be equivalent if one can be transformed into the other by some group element. In our investigation the symmetry group G_1 and G_α reflects the linearity of the Helmholtz equation, we can add solution and multiply them by constants. The group G_2, G_3 and G_4 are space translation symmetry groups of the equation, reflecting the fact that the Helmholtz equation has constant coefficients. The groups G_5, G_6 and G_7 represents well known rotational symmetry groups.

Conflict of Interest

In this manuscript the authors declare that there is no conflict of interest.

References

i. Bao, G., Wei, G. W., & Zhao, S. (2004). Numerical solution of the Helmholtz equation with high wavenumbers. *International Journal for Numerical Methods in Engineering*, *59*(3), 389–408. https://doi.org/10.1002/nme.883
ii. Bollhöfer, M., Grote, M. J., & Schenk, O. (2009). Algebraic Multilevel Preconditioner for the Helmholtz Equation in Heterogeneous Media. *SIAM Journal on Scientific Computing*, *31*(5), 3781–3805. https://doi.org/10.1137/080725702

iii. El-Tantawy, S. A., Salas, A. H., & Alharthi, M. R. (2021). A new approach for modelling the damped Helmholtz oscillator: applications to plasma physics and electronic circuits. *Communications in Theoretical Physics*, *73*(3), 035501. https://doi.org/10.1088/1572-9494/abda1b
iv. Erlangga, Y. A., Vuik, C., & Oosterlee, C. W. (2004). On a class of preconditioners for solving the Helmholtz equation. *Applied Numerical Mathematics*, *50*(3–4), 409–425. https://doi.org/10.1016/j.apnum.2004.01.009
v. Gupta, V. G., & Pal, K. (2013). Harmonic wave excitation in a semi-infinite medium. *Contemporary Engineering Sciences*, *6*, 239–243. https://doi.org/10.12988/ces.2013.211
vi. Hirtum, A. V. (2017). Quasi-analytical solution of two-dimensional Helmholtz equation. *Applied Mathematical Modelling*, *47*, 96–102. https://doi.org/10.1016/j.apm.2017.03.026
vii. Li, W., Wang, Z., Cui, T., Null, Y. X., & Xiang, X. (2023). Deep Domain Decomposition Methods: Helmholtz Equation. *Advances in Applied Mathematics and Mechanics*, *15*(1), 118–138. https://doi.org/10.4208/aamm.OA-2021-0305
viii. Olver, P. J. (1979). Symmetry groups and group invariant solutions of partial differential equations. *Journal of Differential Geometry*, *14*(4). https://doi.org/10.4310/jdg/1214435234
ix. Olver, P. J. (1993). *Application of Lie groups to differential equations*. 2nd Edition, Springer-Verlag. New York, USA.
x. Pal, K., Shekhawat, A. S., & Pawar, V. (2022). Applications of Lie group in two - dimensional heat equation. *SGVU International Journal of Convergence of Technology and Managemen*, *8*(2), 78–83. https://shorturl.at/ryFS9 .
xi. Pal, K., Gupta, V. G., & Pawar, V. (2023). Group analysis for Klein-Gordon equation via their symmetries. *Jñānābha*, 53(1).
xii. Pozer, D. M. (2011). *Microwave Engineering*. 4th edition, John Wiley and Sons, Inc.
xiii. Sakkaravarthi, K., Johnpillai, A. G., Durga Devi, A., Kanna, T., & Lakshmanan, M. (2018). Lie symmetry analysis and group invariant solutions of the nonlinear Helmholtz equation. *Applied Mathematics and Computation*, *331*, 457–472. https://doi.org/10.1016/j.amc.2018.03.011
xiv. Shekhawat, A. S., Pal, K., & Pawar, V. (2022). Wave propagation for vibrating uniform membrane. *SGVU International Journal of Convergence of Technology and Management*, *8*(1), 47–50. https://shorturl.at/apv35
xv. Shiralashetti, S. C., Deshi, A. B., & Kantli, M. H. (2020). The Numerical Solution of Helmholtz Equation Using Modified Wavelet Multigrid Method. *International Journal of Modern Mathematical Sciences*, 18(1), 76-91. https://shorturl.at/gJQR9